# University of Saskatchewan
## Department of Mathematics and Statistics
### Math 223 (05, G.Patrick)

November 21, 2005 — Test #2 — 90 minutes

**This examination consists of two parts. Part A contains short, routine questions, which you should answer fully but succinctly in the space provided. The questions in Part B are more difficult, and some are designed to challenge you. Fully answer all questions of Part B in the space provided.**

**You should complete Part A rapidly, and save at least half your time to answer the questions in Part B. Part A is worth 30 points and Part B is worth 20 points. Remember to print your name and student ID in the spaces provided in both Part A and Part B.**

**The points for each problem are indicated in the right margin.**

**Permitted resources: none. No books, no notes of any kind, no calculators, no electronic devices of any kind.**

**This is a midterm test. Cheating on an test is considered a serious offense by the University and can be met with disciplinary action, including suspension or expulsion. Candidates shall not bring into the test room any books, resources or papers except at the discretion of the examiner or as indicated on the examination paper. Candidates shall hold no communication of any kind with other candidates within the examination room.**

**Print your name and student ID here:** ______________________

---

**PART A. Fully answer the following questions in the space provided.**

*Question A1.* Using the method of Lagrange multipliers, find all critical points of the function $f(x,y,z) = x - y + z$ subject to the constraint $x^2 - 6xy + y^2 - z = 0$. 3

$\nabla f = (1,-1,1) = \lambda \nabla g = \lambda(2x-6y, -6x+2y, -1)$. From the $3^{rd}$ component $\lambda = -1$ and thus $2x-6y=-1$, $6x-2y=-1$

$$x = \frac{\begin{vmatrix} -1 & -6 \\ -1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -6 \\ 6 & -2 \end{vmatrix}} = \frac{2-6}{-4+36} = \frac{-4}{32} = \frac{-1}{8} \qquad y = \frac{\begin{vmatrix} 2 & -1 \\ 6 & -1 \end{vmatrix}}{32} = \frac{4}{32} = \frac{1}{8}$$

$z = x^2 - 6xy + y^2 = \frac{1}{64} + 6\cdot\frac{1}{64} + \frac{1}{64} = \frac{8}{64} = \frac{1}{8}$. So $\left(\frac{-1}{8}, \frac{1}{8}, \frac{1}{8}\right)$ is the only critical point.

*Question A2.* Using differentials, estimate the value of $\ln(1+\sin(2x+y))$ when $x = .1$ and $y = -.1$. 3

$z = \ln(1+\sin(2x+y))$

$dz = \frac{1}{1+\sin(2x+y)} 2\cos(2x+y)\,dx + \frac{1}{1+\sin(2x+y)} \cos(2x+y)\,dy$

If $x=0, y=0, \Delta x = .1, \Delta y = -.1$ then

$\Delta z \approx 2\Delta x + \Delta y = 2(.1) - .1 = .1$

At $x=0, y=0$, $z = \ln(1) = 0$. So $z(.1,-.1) \approx z(0,0) + \Delta z$
$\approx 0 + .1 = .1$

*Question A3.* Calculate, up to and including terms of order 2, the Taylor series of the function $z = \dfrac{1+\sin y}{x}$ at $(x, y) = (1, \pi)$. 3

$$\frac{\partial z}{\partial x}(1,\pi) = -\frac{(1+\sin y)}{x^2}\Big|_{x=1,y=\pi} = -1 \qquad \frac{\partial z}{\partial y}(1,\pi) = \frac{\cos y}{x}\Big|_{x=1,y=\pi} = -1$$

$$\frac{\partial^2 z}{\partial x^2}(1,\pi) = \frac{2(1+\sin y)}{x^3}\Big|_{x=1,y=\pi} = 2 \qquad \frac{\partial^2 z}{\partial x \partial y}(1,\pi) = -\frac{\cos y}{x^2}\Big|_{x=1,y=\pi} = 1$$

$$\frac{\partial^2 z}{\partial y^2}(1,\pi) = -\frac{\sin y}{x}\Big|_{x=1,y=\pi} = 0 \qquad 1-(x-1)-(y-\pi)+\frac{1}{2}\left(2(x-1)^2+2(x-1)(y-\pi)\right)$$

*Question A4.* Calculate the value of the double integral $\iint_R xy^2\,dA$ where $R$ is the region 3

$$R = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 2\}$$

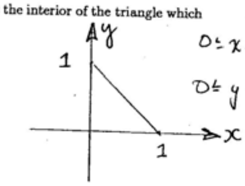
$$\int_0^1\int_0^2 xy^2\,dy\,dx = \int_0^1 x\left(\frac{1}{3}y^3\Big|_{y=0}^{y=2}\right)dx = \frac{8}{3}\int_0^1 x\,dx = \frac{8}{3}\times\frac{1}{2} = \frac{4}{3}$$

*Question A5.* Calculate the iterated double integral $\int_0^1\int_0^y (1+xy)\,dx\,dy$. 3

$$\int_0^1\int_0^y 1+xy\,dx\,dy = \int_0^1 x+\frac{1}{2}x^2y\Big|_{x=0}^{y} = \int_0^1 y+\frac{1}{2}y^3\,dy$$

$$= \frac{1}{2}+\frac{1}{8} = \frac{5}{8}$$

*Question A6.* Calculate the double integral $\iint_R xy\,dA$ where $R$ is the interior of the triangle which has as vertices the three points $(0,0)$, $(1,0)$, and $(0,1)$. 3




$0 \le x \le 1$

$0 \le y \le 1-x$

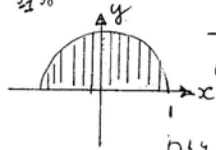
$$\iint_R xy\,dA = \int_0^1\int_0^{1-x} xy\,dy\,dx$$

$$= \int_0^1 \frac{1}{2}xy^2\Big|_{y=0}^{y=1-x}dx = \int_0^1 \frac{1}{2}x(1-x)^2$$

$$= \frac{1}{2}\int_0^1 x-2x^2+x^3\,dx = \frac{1}{2}\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right) = \frac{1}{2}\left(\frac{6-8+3}{12}\right) = \frac{1}{24}$$

*Question A7.* Reverse the order of the iterated double integral $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sin(e^{x^2+y^2})\,dy\,dx$. Do not evaluate the integral. 3

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sin\left(e^{x^2+y^2}\right)\,dy\,dx$$




$-1 \le x \le 1$

$0 \le y \le \sqrt{1-x^2}$

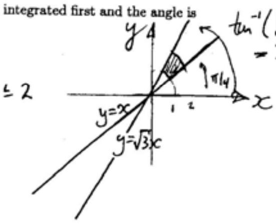
$$= \int_{0}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sin\left(e^{x^2+y^2}\right)\,dx\,dy$$

$0 \le y \le 1$

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$

*Question A8.* Let $R$ be the region in the first quadrant $x \ge 0, y \ge 0$, between the lines $y = x$ and $y = \sqrt{3}x$ and between the circles $x^2+y^2 = 1$ and $x^2+y^2 = 4$. Using polar coordinates, write the integral $\iint_R \ln(x+y)\,dA$ as an iterated integral where the radius is integrated first and the angle is integrated last. Do not evaluate the integral. 3

$$\iint_R \ln(x+y)\,dA = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\int_{1}^{2} \ln\left(r\cos\theta + r\sin\theta\right) r\,dr\,d\theta \qquad 1 \le r \le 2$$




$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

*Question A9.* Calculate, but do not evaluate, a double integral corresponding to the area of the surface of $z = \ln(xy^2)$ above the region $\{(x,y) \mid 1 \le x \le 2, 1 \le y \le 2\}$. 3

$$z = \ln(xy^2) = \ln x + 2\ln y \qquad \frac{\partial z}{\partial x} = \frac{1}{x} \qquad \frac{\partial z}{\partial y} = \frac{2}{y}$$

$$S = \int_{1}^{2}\int_{1}^{2} \sqrt{1 + \frac{1}{x^2} + \frac{4}{y^2}}\,dx\,dy = \int_{1}^{2}\int_{1}^{2} \sqrt{1 + \frac{1}{x^2} + \frac{4}{y^2}}\,dy\,dx$$

*Question A10.* Calculate the triple integral $\iiint_R (x + 2yz)\,dV$ where $R$ is the region 3

$$R = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$$

$$\int_0^1\int_0^1\int_0^1 x + 2yz\,dx\,dy\,dz = \int_0^1\int_0^1 \frac{1}{2}x^2 + 2xyz \Big|_{x=0}^{x=1} dy\,dz$$

$$= \int_0^1\int_0^1 \frac{1}{2} + 2yz\,dy\,dz = \int_0^1 \frac{y}{2} + y^2 z \Big|_{y=0}^{y=1} dz = \int_0^1 \frac{1}{2} + z\,dz$$

$$= \frac{z}{2} + \frac{1}{2}z^2 \Big|_0^1 = \underline{1}$$

Math 223 Test #~~1~~ 2 PART B. (05, G.Patrick)

**PART B. Fully answer the following questions in the space provided.**

**Print your name and student ID here:** ____________________

---

*Question B1.* Calculate the value of the iterated integral 5

$$\int_0^1 \int_{y^2}^{y} \int_0^{x+xy} \frac{xz}{1+y}\, dz\, dx\, dy.$$

$$= \int_0^1 \int_{y^2}^{y} \frac{xz^2}{2(1+y)} \Bigg|_{z=0}^{x+xy} dx\, dy = \int_0^1 \int_{y^2}^{y} \frac{x(x+xy)^2}{2(1+y)}\, dx\, dy$$

$$= \int_0^1 \int_{y^2}^{y} \frac{x^3(1+y)^2}{2(1+y)}\, dx\, dy = \int_0^1 \frac{1+y}{2} \frac{1}{4} x^4 \Bigg|_{x=y^2}^{x=y} dy$$

$$= \int_0^1 \frac{1+y}{8} (y^4 - y^8)\, dy = \frac{1}{8} \int_0^1 y^4 - y^8 + y^5 - y^9\, dy$$

$$= \frac{1}{8}\left(\frac{1}{5} - \frac{1}{9} + \frac{1}{6} - \frac{1}{10}\right) = \frac{1}{8}\left(\frac{18 - 10 + 15 - 9}{90}\right)$$

$$= \frac{1}{8}\left(\frac{14}{90}\right) = \frac{7}{360}$$

*Question B2.* Find the maximum and minimum of the function $f(x, y) = x + \quad y$ on the region bounded by the ellipse $x = \cos t, y = \sqrt{3}\sin t$, $0 \le t \le 2\pi$. 5

$\frac{\partial f}{\partial x} = 1 \neq 0$ so no critical points. Hence max/min occur on the boundary.

$f(\cos t, \sqrt{3}\sin t) = \cos t + \sqrt{3}\sin t$ ; $\frac{df}{dt} = -\sin t + \sqrt{3}\cos t = 0$

$\sin t = \sqrt{3}\cos t$ $\quad t = \frac{\pi}{3}, \frac{4\pi}{3}$

| | $\cos t + \sqrt{3}\sin t$ |
|---|---|
| $\frac{\pi}{3}$ | $\frac{1}{2} + \sqrt{3}\,\frac{\sqrt{3}}{2} = 2$ |
| $\frac{4\pi}{3}$ | $-2$ |

Max is 2
Min is $-2$

*Question B3.* Using the method of Lagrange multipliers, find all critical points of the function $f(x,y,z) = 12x - 2y + 2z$ subject to the constraints $x^2 + z^2 = 2$ and $x^2 + y^2 = 2$. 5

$\nabla f = (12,-2,2) = \lambda \nabla(x^2+z^2-2) + \mu \nabla(x^2+y^2-2) = \lambda(2x,0,2z) + \mu(2x,2y,0)$

$\lambda x + \mu x = 6, \quad \mu y = -1, \quad \lambda z = 1, \quad x^2+z^2=2, \quad x^2+y^2=2$

From $\mu y = -1$, $\lambda z = 1$ we know $\mu \neq 0$, $\lambda \neq 0$ so

$x(\lambda+\mu) = 6, \quad x^2 + \frac{1}{\lambda^2} = 2, \quad x^2 + \frac{1}{\mu^2} = 2 \quad \Rightarrow \frac{1}{\lambda^2} - \frac{1}{\mu^2} = 0 \quad \Rightarrow \lambda = \pm\mu$

But $x(\lambda+\mu) = 6$ so $\lambda+\mu \neq 0$ so $\lambda = -\mu$ is not possible. Thus, $\lambda = \mu$.

$2x\lambda = 6, \quad x^2 + \frac{1}{\lambda^2} = 2 \quad \Rightarrow \quad \frac{9}{\lambda^2} + \frac{1}{\lambda^2} = 2 \quad \Rightarrow \quad \lambda = \pm\sqrt{5} = \mu$

$y = \frac{-1}{\mu} = \mp\frac{1}{\sqrt{5}}, \quad z = \frac{1}{\lambda} = \pm\frac{1}{\sqrt{5}}, \quad x = \frac{6}{\lambda+\mu} = \frac{\pm 6}{2\sqrt{5}} = \frac{\pm 3}{\sqrt{5}}$

So there are two critical points: $\left(\frac{3}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ $\left(\frac{-3}{\sqrt{5}}, \frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$

*Question B4.* Find the value of the triple integral $\iiint_R 32xyz\,dV$ where $R$ is the region inside the unit sphere centered at the origin and in the first octant. Use cartesian coordinates ? 5

$R$: $0 \le x \le 1$
$0 \le y \le \sqrt{1-x^2}$
$0 \le z \le \sqrt{1-x^2-y^2}$

$\iiint_R 32xyz\,dV = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 32xyz\,dz\,dy\,dx$

$= \int_0^1 \int_0^{\sqrt{1-x^2}} 16xyz^2 \Big|_{z=0}^{z=\sqrt{1-x^2-y^2}} dy\,dx = \int_0^1 \int_0^{\sqrt{1-x^2}} 16xy(1-x^2-y^2)\,dy\,dx$

$= \int_0^1 8x(1-x^2)y^2 - 4xy^4 \Big|_{y=0}^{\sqrt{1-x^2}} dx = \int_0^1 8x(1-x^2)(1-x^2) - 4x(1-x^2)^2 dx$

$= \int_0^1 4x(1-2x^2+x^4) = 4 \cdot \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{2}{3}$